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PERTURBATION OF AN EIGEN-VALUE FROM A DENSE POINT SPECTRUM : AN EXAMPLE

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Abstract

We study a perturbed Floquet Hamiltonian $K + \beta V$ depending on a coupling constant β . The spectrum $\sigma(K)$ is assumed to be pure point and dense. We pick up an eigen-value, namely $0 \in \sigma(K)$, and show the existence of a function $\lambda(\beta)$ defined on $I \subset \mathbb{R}$ such that $\lambda(\beta) \in \sigma(K + \beta V)$ for all $\beta \in I$, 0 is a point of density for the set I , and the Rayleigh-Schrödinger perturbation series represents an asymptotic series for the function $\lambda(\beta)$. All ideas are developed and demonstrated when treating an explicit example but some of them are expected to have an essentially wider range of application.

Key-Words: Floquet Hamiltonians, dense point spectrum, perturbation theory

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1 Introduction

A common problem occurring frequently in theoretical physics is the eigen-value problem for a perturbed operator $K + \beta V$, with β being a coupling constant, under the assumption that F_0 is a known eigen-value of the unperturbed operator K . The Rayleigh-Schrödinger (RS) series gives a formal solution $F(\beta)$, with $F(0) = F_0$, as an unambiguously determined formal power series. The regular perturbation theory due to Rellich (1937) and Kato (1966) justifies this formal series as an analytic function well defined on a neighbourhood of $\beta = 0$ provided one essential condition is fulfilled – the eigen-value $F_0 \in \sigma(K)$ must be *isolated*. On the other hand, the situation when an eigen-value of K is not isolated is far away of being exceptional and recently attracted a considerable attention (see Simon 1993 and references therein).

So called Floquet Hamiltonians represent a class of operators having even a dense pure point spectrum in many interesting examples. They were introduced as an important tool to study time-dependent systems (see Howland 1979, Yajima 1977). A distinguished subclass is formed by the systems with the potential $V(t)$ being T -periodic and bounded. The period is usually considered as a parameter. After rescaling the time, the potential $V(t)$ becomes 2π -periodic and the frequency $\omega = 2\pi/T$ appears in front of the time derivative. Thus one is lead to study the operator $K + \beta V(t)$ acting in

$\mathcal{K} := L^2(\mathbb{T}, dt) \otimes \mathcal{H}$, with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and

$$K := -i\omega\partial_t + H, \quad \omega > 0,$$

where H is the "true" Hamiltonian acting as a self-adjoint operator in a separable Hilbert space \mathcal{H} . We use the loose notation identifying ∂_t with $\partial_t \otimes 1$, H with $1 \otimes H$ etc. Provided the spectrum $\sigma(H)$ is pure point the same is true for $\sigma(K) = \omega\mathbb{Z} + \sigma(H)$. It is known that $\sigma(K)$ is dense in \mathbb{R} for almost all $\omega > 0$ as soon as $\sup \sigma(H) = +\infty$. Recently the spectrum of $K + \beta V(t)$ has been studied by the aid of a quantum version of the KAM method due to Bellissard (1983) (see also Combescure 1987, Bellissard, Vittot 1990, Bleher, Jauslin, Lebowitz 1992, Duclos, Šťovíček 1996) as well as by adiabatic tools (Nenciu 1993, Joye 1994).

In the present paper we focus on a particular example with $\mathcal{H} = L^2(\mathbb{T}, dx)$,

$$H = -\partial_x^2 \text{ (+ periodic boundary conditions), } V(t) = 4 \cos t \cos x. \quad (1)$$

Clearly, $\sigma(H) = \{E(k) = k^2; k \in \mathbb{Z}\}$ and so $\sigma(K) = \{F(n) = \omega n_1 + E(n_2); n \in \mathbb{Z} \times \mathbb{Z}\}$. The spectrum of H is degenerate and that makes the problem more complicated; the only non-degenerate eigen-value is $E(0) = 0$. This is why we restrict ourselves to eigen-values $F(n)$ of K with $n_2 = 0$. In order to be specific, we shall even consider the only eigen-value $F(0) = 0$. We are going to address the question whether there exists an eigen-value $\lambda(\beta)$ of the operator $K + \beta V(t)$ which could be considered as a perturbation of $F(0) = 0$ depending on the parameter β . A possible answer is given in

Proposition 1. *For almost all $\omega > 0$, there exists a real-valued function $\lambda(\beta)$ defined on $I \subset \mathbb{R}$ with the properties:*

- (1) *for $\forall \beta \in I$, $\lambda(\beta)$ is an eigen-value of $K + \beta V(t)$,*
- (2) *$\lim_{\delta \downarrow 0} |I \cap [-\delta, \delta]|/2\delta = 1$,*
- (3) *the function $\lambda(\beta)$ has an asymptotic expansion at $\beta = 0$ coinciding with the formal Rayleigh-Schrödinger perturbation series for the eigen-value $F(0) = 0$ of K .*

In fact, our final goal (not achieved in this paper) is to prove a similar proposition for a much wider class of Floquet Hamiltonians. However, as this program seems to be extremely complex, we preferred to develop and to demonstrate the main ideas when treating an explicit example. But the proof, even in the case of our very particular model, is far away of being

obvious and straightforward. We note that the essential assumptions which are expected to be required also in the general case are a sufficient smoothness of $V(t)$ (generally the order of the asymptotic series depends on the order of differentiability of $V(t)$) and a gap condition imposed on the eigen-values of H : $\sigma(H) = \{E(k); k \in \mathbb{Z}_+\}$ and

$$\inf_{k \in \mathbb{Z}_+} \frac{E(k+1) - E(k)}{(k+1)^\alpha} =: C_E > 0 \quad \text{for some } \alpha > 0, \quad (2)$$

(basically $\alpha = 1$ in our example when overlooking the degeneracy). Apparently, our model captures already all basic features but, on the other hand, it makes possible some simplifications and can be treated on a relatively elementary level. The rest of the paper is devoted to the proof of Proposition 1 but, whenever possible, we shall try to consider a more general situation and to propose some ideas applicable also to other models.

2 Basic equation

The starting point is the eigen-value equation for $K + \beta V$. Assume that 0 is a non-degenerate eigen-value of K and f is the normalized eigen-vector. Let P be the orthogonal projector onto the eigen-space $\mathbb{C}f$ and $Q := 1 - P$. We are seeking $\lambda = \lambda(\beta) \in \mathbb{R}$ and $g \in \mathcal{K}$ such that $Pg = 0$ and

$$(K + \beta V)(f + g) = \lambda(f + g). \quad (3)$$

Without loss of generality we can assume that

$$PVP = 0. \quad (4)$$

Apply successively the projectors P and Q to the equation (3). The result is

$$\lambda = \beta \langle Vf, g \rangle, \quad (5)$$

$$(\hat{K} + \beta \hat{V} - \lambda)g = -\beta QVf. \quad (6)$$

Here and everywhere in what follows the hat indicates the restriction to $\text{Ran } Q$ in the sense: $\hat{X} = QXQ|_{\text{Ran } Q}$.

According to our assumptions, \hat{K} is invertible and we set $\Gamma_0 := \hat{K}^{-1}$ (defined on $\text{Ran } Q$). For $\lambda \notin \sigma(\hat{K})$ we define also

$$\Gamma_\lambda := (\hat{K} - \lambda)^{-1} = (1 - \lambda \Gamma_0)^{-1} \Gamma_0.$$

Keeping λ as an auxiliary parameter one can solve formally (6)

$$g = g(\beta, \lambda) := -\beta(1 + \beta\Gamma_\lambda\hat{V})^{-1}\Gamma_\lambda QVf. \quad (7)$$

Plugging (7) into (5) we get a fixed-point equation for the eigen-value $\lambda = \lambda(\beta)$,

$$\begin{aligned} \lambda &= G(\beta, \lambda) \quad \text{where} \\ G(\beta, \lambda) &:= -\beta^2 \langle QVf, (1 + \beta\Gamma_\lambda\hat{V})^{-1}\Gamma_\lambda QVf \rangle. \end{aligned} \quad (8)$$

The trick with the projectors and keeping λ as an auxiliary parameter is well known and related to various names. In the regular case, when $d := \text{dist}(0, \sigma(\hat{K})) > 0$, one can rederive this way Rellich-Kato Theorem. Indeed, we have $\|\Gamma_0\| = d^{-1}$ and $(1 + \beta\Gamma_\lambda\hat{V})$ is invertible (on $\text{Ran } Q$) provided $|\beta|$ and $|\lambda|$ are sufficiently small. The implicit function theorem applied to (8) then gives the result.

To solve (8) formally one can use B rmann-Lagrange Formula which can be proven with some combinatorics and not necessarily with the Cauchy Residuuum Theorem. Write

$$\begin{aligned} G(\beta, \lambda) &= \sum_{M=0}^{\infty} \Phi_M(\beta) \lambda^M, \quad \text{where} \\ \Phi_M(\beta) &= -\sum_{k=1}^{\infty} \sum_{\mu \in \mathbb{N}^k, |\mu|=k+M} (-\beta)^{k+1} \langle QVf, \hat{K}^{-\mu_1} \hat{V} \hat{K}^{-\mu_2} \dots \hat{V} \hat{K}^{-\mu_k} QVf \rangle. \end{aligned}$$

The formal solution $\lambda(\beta)$ reads

$$\lambda(\beta) = \sum_{N=1}^{\infty} \sum_{\nu \in \mathcal{T}(N)} \Phi_{\nu_1}(\beta) \dots \Phi_{\nu_N}(\beta) = \sum_{M=2}^{\infty} \xi_M \beta^M, \quad (9)$$

where $\mathcal{T}(N) \subset \mathbb{Z}_+^N$ is the set of rooted N -trees: $\nu = (\nu_1, \dots, \nu_N) \in \mathcal{T}(N)$ iff $\nu_k + \dots + \nu_N \leq N - k$, $2 \leq k \leq N$, and $|\nu| = N - 1$. Consequently, one gets an expression for the coefficients ξ_M

$$\begin{aligned} \xi_M &= \sum_{N=1}^{[M/2]} \sum_{\nu \in \mathcal{T}(N)} \sum_{k(1), \dots, k(N) \in \mathbb{N}} \sum_{\mu(1) \in \mathbb{N}^{k(1)}, \dots, \mu(N) \in \mathbb{N}^{k(N)}} \\ &\quad \times (-1)^{M+N} \prod_{j=1}^N \langle QVf, \hat{K}^{-\mu(j)_1} \hat{V} \hat{K}^{-\mu(j)_2} \dots \hat{V} \hat{K}^{-\mu(j)_{k(j)}} QVf \rangle, \end{aligned} \quad (10)$$

with the summation range being restricted by

$$k(1) + \dots + k(N) + N = M, \quad \text{and} \quad |\mu(j)| = k(j) + \nu_j, \quad 1 \leq j \leq N.$$

Of course, this result must coincide with the standard RS perturbation series written in the form (see Kato 1966)

$$\xi_M = \frac{(-1)^M}{M} \sum_{k_1 + \dots + k_M = M-1, \, k_i \geq 0} \text{tr} \left(V \hat{R}^{k_1} \dots V \hat{R}^{k_M} \right), \quad (11)$$

where the symbol \hat{R}^k is defined by: $\hat{R}^0 = -P$, and for $k \geq 1$, $\hat{R}^k|_{\text{Ran } P} = 0$, $\hat{R}^k|_{\text{Ran } Q} = \hat{K}^{-k}$. The equality between (10) and (11) can be verified quite straightforwardly using (4) and the following fact:

Lemma 2. *For a given $N \in \mathbb{N}$ and each $\sigma = (\sigma_1, \dots, \sigma_N) \in \mathbb{Z}_+^N$ obeying $|\sigma| = N - 1$ there exists exactly one cyclic permutation of σ , $\sigma' = (\sigma_{N-m+1}, \dots, \sigma_N, \sigma_1, \dots, \sigma_{N-m})$ (determined by $m \in \{0, 1, \dots, N-1\}$), such that $\sigma' \in \mathcal{T}(N)$.*

Hence each term of (10) is a grouping of many terms of (11) where we take into account the cyclic property of the trace.

However in the case when $\sigma(K)$ is dense in \mathbb{R} and so $\text{dist}(0, \sigma(\hat{K})) = 0$ it seems to be hopeless to consider the RS series as a convergent series. The complication comes from arbitrarily large powers of \hat{K}^{-1} in (10) (or (11)) since among eigen-values of \hat{K} there are arbitrarily small numbers – so called small denominators. Probably the maximum one can attempt in this situation is to verify the finiteness of the coefficients ξ_M (generally up to some order depending on the smoothness of $V(t)$) and to show that the RS series is asymptotic for the function $\lambda(\beta)$.

Let us specify the formula (10) to our example (1). Consider $V(t)$ as an operator in \mathcal{K} and denote by $V(m, n)$, $m, n \in \mathbb{Z}^2$, its matrix elements in the eigen-basis of K . We have

$$V(m, n) = \begin{cases} 1 & \text{if } m - n \in \{\pm(1, 1), \pm(1, -1)\}, \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

Concerning the eigen-values of K , there is a degeneracy

$$F(n_1, n_2) = F(n_1, -n_2) = \omega n_1 + n_2^2.$$

Let $\mathbb{L} = \mathbb{Z}(1, 1) + \mathbb{Z}(1, -1)$ be a sublattice in \mathbb{Z}^2 and denote by $\mathcal{P}_0(N) \subset (\mathbb{Z}^2)^{N+1}$ the set of closed paths in \mathbb{L} of length N with the base point $\bar{0}$: $(\bar{\iota}(0), \bar{\iota}(1), \dots, \bar{\iota}(N)) \in \mathcal{P}_0(N)$ iff $\bar{\iota}(0) = \bar{\iota}(N) = \bar{0}$, $\bar{\iota}(j) \neq \bar{0}$ for $1 \leq j \leq N-1$, $\bar{\iota}(j) - \bar{\iota}(j-1) \in \{\pm(1, 1), \pm(1, -1)\}$ for $1 \leq j \leq N$. Note that $\mathcal{P}_0(N) = \emptyset$ for N odd. Clearly,

$$\langle QVf, \hat{K}^{-\mu_1} \hat{V} \hat{K}^{-\mu_2} \dots \hat{V} \hat{K}^{-\mu_k} QVf \rangle = \sum_{\bar{\iota} \in \mathcal{P}_0(k+1)} \prod_{j=1}^k F(\bar{\iota}(j))^{-\mu_j}. \quad (13)$$

The only thing we can claim at this moment is that all ξ_M , $2 \leq M$, are finite for the sum on the RHS of (10) is finite.

3 Diophantine estimates

In order to cope with small denominators we need diophantine estimates. Suppose that we are given two sequences ψ and E such that

$$\psi : \mathbb{N} \rightarrow]0, 1/2], \quad \sum_{k \in \mathbb{N}} \psi(k) < \infty,$$

and

$$E : \mathbb{N} \rightarrow]0, +\infty[, \quad \inf_{k \in \mathbb{N}} E(k) =: d_E > 0.$$

Set $F(n) := \omega n_1 + E(n_2)$, $n \in \mathbb{Z} \times \mathbb{N}$, and to a constant $\gamma > 0$ relate the set

$$\Omega(\gamma) := \{\omega > 0; \forall n \in \mathbb{Z} \times \mathbb{N}, |F(n)| \geq \omega \gamma \psi(n_2)\}.$$

It is quite standard to show

Lemma 3. *If $\gamma \leq d_E/a \leq 1$ then*

$$|]0, a] \setminus \Omega(\gamma)| \leq \left(16a \sum_{k \in \mathbb{N}} \psi(k) \right) \gamma.$$

We can now introduce the set Ω (depending on ψ) of "non-resonant" frequencies,

$$\Omega := \{\omega > 0; \inf_{n \in \mathbb{Z} \times \mathbb{N}} |F(n)|/\psi(n_2) > 0\} = \bigcup_{\gamma > 0} \Omega(\gamma).$$

As an immediate consequence of Lemma 3 we have

Lemma 4. *The complement $]0, +\infty[\setminus \Omega$ is of zero measure in the Lebesgue sense.*

In the case of our model, $E(k) = k^2$. Extend the definition of ψ by $\psi(0) = 1$ and we define also $F((k, 0)) := \omega k$. We fix once for all $\omega \in \Omega$ (and we don't emphasize this fact anymore in the rest of the paper). Then there exists γ , $0 < \gamma \leq 1$, such that

$$|F(n)| \geq \omega \gamma \psi(|n_2|), \quad \forall n \neq \bar{0}.$$

Rather than treating the formal RS series (9) we wish to attack the fixed-point equation (8). This means to cope with expressions involving the operator Γ_λ and hence the numbers $(F(n) - \lambda)^{-1}$ – the eigen-values of Γ_λ . The estimate on $F(n) - \lambda$ will be governed by a constant ρ and a sequence $\tilde{\psi}$ of positive reals and we require

$$\rho \in [0, 1] \quad \text{and} \quad \tilde{\psi}(k) \leq \psi(k)/2, \quad \forall k \in \mathbb{Z}_+.$$

For a given sequence E as above we define a set Λ of "good" parameters λ ,

$$\Lambda := \{\lambda \in \mathbb{R}; \forall n \in \mathbb{Z} \times \mathbb{N}, |F(n) - \lambda| \geq \omega \gamma (2|\lambda|/\omega)^\rho \tilde{\psi}(|n_2|)\}; \quad (14)$$

note that $|F(n) - \lambda| \geq \omega/2$ for $n_1 \neq 0$, $n_2 = 0$ and $|\lambda| \leq \omega/2$. The following lemma is also easy to prove:

Lemma 5. *If $0 < \delta \leq 1/4$ then*

$$|[-\delta\omega, \delta\omega] \setminus \Lambda| < 2\omega(2\delta)^\rho \sum_{k \in \mathbb{N}, \psi(k) < 2\delta} \tilde{\psi}(k).$$

The standard choice for ψ and $\tilde{\psi}$ is

$$\psi(k) = k^{-\sigma}/2, \quad \tilde{\psi}(k) = k^{-\tau}/4, \quad \text{with } 1 < \sigma \leq \tau. \quad (15)$$

In this case we get another intermediate result as a direct consequence of Lemma 5.

Lemma 6. *If $\tau > 1 + \sigma(1 - \rho)$ then 0 is a point of density for the set Λ , i.e.,*

$$\lim_{\delta \downarrow 0} \frac{1}{2\delta\omega} |[-\delta\omega, \delta\omega] \cap \Lambda| = 1.$$

Suppose that the sequence E obeys the gap condition (2) with $\alpha > 0$. A possible choice of the constants σ , τ and ρ which suits the assumption of Lemma 6 is

$$\tau = 1 + \alpha, \quad 1 < \sigma < 1 + \alpha, \quad \text{and} \quad \rho = 1/\sigma.$$

In our model we have effectively $\alpha = 1$ and so we choose

$$\tau = 2, \quad 1 < \sigma < 2, \quad \text{and} \quad \rho = 1/\sigma \in]1/2, 1[. \quad (16)$$

Let us now derive some consequences of the above diophantine estimates in combination with the gap condition (2). Suppose again that the spectrum of H is pure point and equals $\{E(k)\}_{k \in \mathbb{Z}_+}$, $E(0) = 0$, and that E obeys the gap condition (2). It is quite useful to observe that another inequality follows straightforwardly from (2),

$$|E(j) - E(k)| \geq \frac{C_E}{\alpha + 1} |j - k| \max\{j^\alpha, k^\alpha\}, \quad \forall j, k \in \mathbb{Z}_+. \quad (17)$$

We shall denote by P_n , $n \in \mathbb{Z} \times \mathbb{Z}_+$ (or $\mathbb{Z} \times \mathbb{Z}$ in our model), the eigenprojectors of K corresponding to the eigen-values $F(n)$; we have $P \equiv P_0$ with $F(0) = 0$. We set also $Q_n := 1 - P_n$.

Another important observation coming from the gap condition is that those eigen-states P_n which can potentially contribute by small denominators are distributed rather rarely in the half-plane $n_2 \geq 0$. Let \mathcal{S} designate the set of "critical" indices defined by:

$$n \in \mathcal{S} \text{ iff } F(n) \in]-\omega/2, \omega/2] \setminus \{0\}. \quad (18)$$

Clearly, to each $n_2 \in \mathbb{N}$ there exists exactly one $n_1 \in \mathbb{Z}$ (necessarily $n_1 \leq 0$) such that $n \in \mathcal{S}$; $(n_1, 0) \notin \mathcal{S}$ for all $n_1 \neq 0$, and we treat $n = \bar{0}$ separately since it corresponds to the eigen-state P to be perturbed. Furthermore, if $m, n \in \mathcal{S}$ and $m_2 \leq n_2$ then $|m_1| \leq |n_1|$. Roughly speaking, the indices from the set \mathcal{S} are situated closely to the curve $n_1 = -E(n_2)/\omega$. We set $P_{\mathcal{S}} := \sum_{n \in \mathcal{S}} P_n$, $Q_{\mathcal{S}} := Q - P_{\mathcal{S}}$. Evidently, $\|\Gamma_0 Q_{\mathcal{S}}\| \leq 2/\omega$.

Let us introduce a function defined on \mathcal{S} ,

$$L(n) := \min\{|n_2|, d(n)\}, \quad (19)$$

with pr_1 being the projection onto the first coordinate axis, and:

$$d(n) := \text{dist}(n_1, \text{pr}_1(\mathcal{S} \setminus \{n\})) = \min_{n' \in \mathcal{S}, |n'_2 - n_2| = 1} |n'_1 - n_1| \leq \text{dist}(n_1, \text{pr}_1(\mathcal{S}) \setminus \{n_1\}).$$

Lemma 7. Assume that the function $\tilde{\psi}$ occurring in the definition (14) of the set Λ satisfies

$$\sup_{k \in \mathbb{N}} k^{-\min\{1, \alpha\}} |\log \tilde{\psi}(k)| < \infty.$$

Then there exists a constant $C_1 > 1$ such that

$$|F(n) - \lambda| \geq (2|\lambda|/\omega)^\rho C_1^{-L(n)} \quad \text{for } \forall n \in \mathcal{S}, \forall \lambda \in \Lambda.$$

Proof. It is sufficient to find C_1 so that

$$\omega \gamma \tilde{\psi}(n_2) \geq \max\{C_1^{-n_2}, C_1^{-d(n)}\},$$

holds for all $n \in \mathcal{S}$. Observe that for any couple $m, n \in \mathcal{S}$, $m \neq n$, we have $m_2 \neq n_2$ and

$$\omega |n_1 - m_1| \geq |E(n_2) - E(m_2)| - |F(n) - \lambda| - |F(m) - \lambda|,$$

and consequently, in virtue of (17) and the definition (18) of \mathcal{S} ,

$$d(n) \geq (C_E/(\alpha + 1)) |n_2|^\alpha - \omega. \quad (20)$$

The rest of the proof is evident. ■

We are going to verify one more estimate related to the function $L(n)$ defined in (19). To this end we shall need

Lemma 8. Let $\Delta_0, \Delta_1, \dots, \Delta_\ell$ be a family of positive numbers. Then it holds

$$\left| \frac{1}{\Delta_1 + \Delta_2 + \dots + \Delta_\ell} - \frac{1}{\ell \Delta_0} \right| \leq \max_{1 \leq k \leq \ell} \left| \frac{1}{\Delta_k} - \frac{1}{\Delta_{k-1}} \right|.$$

Proof. The proof follows immediately from the identity

$$\begin{aligned} \frac{1}{\Delta_1 + \Delta_2 + \dots + \Delta_\ell} - \frac{1}{\ell \Delta_0} &= \frac{1}{\ell} \left[\left(\frac{1}{\Delta_1} - \frac{1}{\Delta_0} \right) (\Delta_1 + \dots + \Delta_\ell) \right. \\ &\quad + \left(\frac{1}{\Delta_2} - \frac{1}{\Delta_1} \right) (\Delta_2 + \dots + \Delta_\ell) \\ &\quad + \dots \\ &\quad \left. + \left(\frac{1}{\Delta_\ell} - \frac{1}{\Delta_{\ell-1}} \right) \Delta_\ell \right] \frac{1}{\Delta_1 + \dots + \Delta_\ell}. \quad \blacksquare \end{aligned}$$

Let us define

$$\Delta E(k) := E(k+1) - E(k), \quad k \in \mathbb{Z}_+,$$

and suppose that E still satisfies the gap condition (2), $E(0) = 0$. Concerning the function $\tilde{\psi}$ we assume that it is decreasing and

$$\sup_{k \in \mathbb{N}} \tilde{\psi}(k/2)/\tilde{\psi}(k) =: C_\psi < \infty. \quad (21)$$

The following lemma contains a condition relating the sequences ΔE and $\tilde{\psi}$.

Lemma 9. *Assume that*

$$\sup_{k \in \mathbb{Z}_+} \frac{1}{\tilde{\psi}(k)} \left| \frac{1}{\Delta E(k+1)} - \frac{1}{\Delta E(k)} \right| =: C_\Delta < \infty. \quad (22)$$

Then there exists a constant $C_2 > 0$ such that for each $n \in \mathcal{S}$ verifying

$$\min\{\Delta E(n_2), \Delta E(n_2 - 1)\} \geq 4\omega, \quad (23)$$

for all $m \in \mathbb{Z} \times \mathbb{N}$, $m \neq n$, from the neighbourhood

$$2 \max\{|n_1 - m_1|, |n_2 - m_2|\} \leq L(n), \quad (24)$$

and for all $\lambda \in \Lambda \cap [-\omega/3, \omega/3]$ it holds true that

$$\left| \frac{1}{F(m) - \lambda} + \frac{1}{F(m') - \lambda} \right| \leq C_2 (2|\lambda|/\omega)^{-\rho} |F(n) - \lambda|,$$

where $m' = 2n - m$.

Proof. The assumptions have some obvious consequences. First,

$$2|n_1 - m_1| \leq \text{dist}(n_1, \text{pr}_1(\mathcal{S} \setminus \{n\})), \text{ and } m \neq n,$$

implies that $m \notin \mathcal{S}$. Thus one finds that

$$|F(m) - \lambda| \geq \left(\frac{1}{2} - \frac{1}{3}\right)\omega = \frac{1}{6}\omega.$$

Obviously, (24) also implies that $n_2/2 \leq m_2 \leq 3n_2/2$.

Furthermore, we have

$$|F(m) - \lambda| \geq |E(m_2) - E(n_2)|/6. \quad (25)$$

Indeed, if $m_2 \neq n_2$ then

$$|F(m) - \lambda| \geq |E(m_2) - E(n_2)| \left(1 - \frac{\omega|m_1 - n_1| + |F(n)| + |\lambda|}{|E(m_2) - E(n_2)|} \right).$$

Let $n' \in \mathcal{S}$ be such that $|n'_2 - n_2| = 1$ and $\text{sgn}(n'_2 - n_2) = \text{sgn}(m_2 - n_2)$. Then $\text{dist}(n_1, \text{pr}_1(\mathcal{S} \setminus \{n\})) \leq |n_1 - n'_1|$ and, owing to (24),

$$\begin{aligned} 2\omega|n_1 - m_1| &\leq \omega|n_1 - n'_1| = |E(n'_2) - E(n_2) + F(n) - F(n')| \\ &\leq |E(m_2) - E(n_2)| + \left(\frac{\omega}{2} + \frac{\omega}{2} \right). \end{aligned}$$

Note that $(m_2 \neq n_2)$

$$|E(m_2) - E(n_2)| \geq \min\{\Delta E(n_2), \Delta E(n_2 - 1)\} \geq 4\omega.$$

Altogether this means that

$$\frac{\omega|m_1 - n_1| + |F(n)| + |\lambda|}{|E(m_2) - E(n_2)|} \leq \frac{1}{2} + \left(\frac{\omega}{2} + \frac{\omega}{2} + \frac{\omega}{3} \right) \frac{1}{4\omega} = \frac{5}{6}$$

and (25) follows. All the above estimates are also valid for m' .

Write now

$$\frac{1}{F(m) - \lambda} + \frac{1}{F(m') - \lambda} = \frac{2(F(n) - \lambda) + E(m_2) + E(m'_2) - 2E(n_2)}{(F(m) - \lambda)(F(m') - \lambda)}.$$

Now to finish the proof, it suffices to study the case $m_2 - n_2 = n_2 - m'_2 \neq 0$. From (25) one finds that

$$\begin{aligned} 6^{-2} \left| \frac{E(m_2) + E(m'_2) - 2E(n_2)}{(F(m) - \lambda)(F(m') - \lambda)} \right| &\leq \left| \frac{1}{E(m_2) - E(n_2)} + \frac{1}{E(m'_2) - E(n_2)} \right| \\ &\leq \left| \frac{1}{E(m_2) - E(n_2)} - \frac{1}{(m_2 - n_2)\Delta E(n_2)} \right| \\ &\quad + \left| \frac{1}{E(m'_2) - E(n_2)} - \frac{1}{(m'_2 - n_2)\Delta E(n_2)} \right| \quad (26) \end{aligned}$$

Combining Lemma 8, the monotone behaviour of $\tilde{\psi}$, and the assumption (22) we get

$$\left| \frac{1}{E(j + \ell) - E(j)} - \frac{1}{\ell \Delta E(j)} \right| \leq C_\Delta \tilde{\psi}(j),$$

$$\left| \frac{1}{E(j) - E(j - \ell)} - \frac{1}{\ell \Delta E(j)} \right| \leq C_{\Delta} \tilde{\psi}(j - \ell).$$

Thus we can estimate from above the RHS of (26) by (c.f. (21))

$$\begin{aligned} 2C_{\Delta} \tilde{\psi}(\min\{m_2, m'_2\}) &\leq 2C_{\Delta} \tilde{\psi}(n_2/2) \leq 2C_{\Delta} C_{\psi} \tilde{\psi}(n_2) \\ &\leq (2C_{\Delta} C_{\psi} / \omega \gamma) (2|\lambda|/\omega)^{-\rho} |F(n) - \lambda|. \end{aligned}$$

This completes the proof. ■

Finally note that, with the choice of $\tilde{\psi}$ (15) and for $E(k) = k^2$, the assumptions of both Lemma 7 and Lemma 8 are satisfied. Thus these two lemmas are applicable to our example provided the choices (15) and (16) have been made.

4 Solution of the fixed-point equation

We wish to justify the power series

$$g(\beta, \lambda) = \sum_{k=0}^{\infty} (-\beta)^{k+1} (\Gamma_{\lambda} \hat{V})^k \Gamma_{\lambda} Q V f \quad (27)$$

as a solution to the vector equation (6). We start from an estimate whose proof relies heavily on the very special features of our model. This doesn't concern the spectrum of H (the gap condition (2) would be sufficient) but what is really special is the form of the potential (12). For each $m \in \mathbb{Z}^2$ there exist exactly four indices $n \in \mathbb{Z}^2$ such that $V_{mn} \neq 0$. This fact makes it possible to use some elementary combinatorics in order to treat the summands in (27). The heart of the proof is a sort of compensation based on Lemma 9. This method of compensations is inspired by the pioneer work of Eliasson (1988).

Recall the definition of the lattice \mathbb{L} (Sec.2) and denote by $\mathcal{P}(N) \subset (\mathbb{Z}^2)^{N+1}$ the set of (unclosed) paths in \mathbb{L} of length N with the initial vertex $\bar{0}$: $(\bar{i}(0), \bar{i}(1), \dots, \bar{i}(N)) \in \mathcal{P}(N)$ iff $\bar{i}(0) = \bar{0}$, $\bar{i}(j) \neq \bar{0}$ for $1 \leq j \leq N$, and $\bar{i}(j) - \bar{i}(j-1) \in \{\pm(1, 1), \pm(1, -1)\}$ for $1 \leq j \leq N$. Clearly, $|\mathcal{P}(N)| \leq 4^N$. For $M \in \mathbb{N}$ one can write

$$(\Gamma_{\lambda} \hat{V})^{M-1} \Gamma_{\lambda} Q V P = \sum_{\bar{i} \in \mathcal{P}(M)} \left(\prod_{j=1}^M \frac{1}{F(\bar{i}(j)) - \lambda} \right) P_{\bar{i}(M)}. \quad (28)$$

Lemma 10. *In the case of the model (1) and assuming that the choices (15) and (16) have been made, there exists a constant $\hat{C} > 0$ such that*

$$\|\Gamma_\lambda QVf\| \leq \hat{C}, \quad \|(\Gamma_\lambda \hat{V})^{M-1} \Gamma_\lambda QVf\| \leq \left(\frac{2|\lambda|}{\omega}\right)^\rho \left(\left(\frac{2|\lambda|}{\omega}\right)^{-\rho/2} \hat{C}\right)^M$$

holds true for $\forall M \in \mathbb{N}$, $M \geq 2$, and $\forall \lambda \in \Lambda \cap [-\omega/3, \omega/3]$, $\lambda \neq 0$.

Remark. Note the type of the estimate: we are able to estimate the vector $(\Gamma_\lambda \hat{V})^{M-1} \Gamma_\lambda QVf$ but not directly the operator $(\Gamma_\lambda \hat{V})^M$.

Proof. We start from restricting the set \mathcal{S} of critical indices to a subset $\mathcal{S}' = \{n \in \mathcal{S}; |n_2| > b\}$. The bound $b \in \mathbb{N}$ is required to obey the conditions:

- $b \geq 3$,
- $4\omega \leq \min\{\Delta(k), \Delta(k-1)\}$ for $\forall k > b$,
- $L(n) \geq 2$ for $\forall n \in \mathcal{S}$, $|n_2| > b$.

The second requirement is dictated by the assumption (23) of Lemma 9 and the third one is possible since from the estimate (20) follows that

$$\lim_{n \in \mathcal{S}, |n_2| \rightarrow \infty} L(n) = +\infty.$$

Clearly, since $|F(n) - \lambda| \geq \omega/6$ for $n \notin \mathcal{S}$, $|\lambda| \leq \omega/3$, there exists a constant $C_3 > 0$ such that

$$|F(n) - \lambda| \geq C_3 \quad \text{for } \forall n \notin \mathcal{S}', \quad \forall \lambda \in \Lambda \cap [-\omega/3, \omega/3].$$

Without loss of generality we can restrict ourselves to $M \geq 2$. For each $\bar{i} \in \mathcal{P}(M)$ the vertices from \mathcal{S}' split the path into segments. Consider such a segment of length ℓ , $(\bar{i}(j), \bar{i}(j+1), \dots, \bar{i}(j+\ell))$, with $\bar{i}(j+\ell) \in \mathcal{S}'$, and also $\bar{i}(j) \in \mathcal{S}'$ provided $j \neq 0$, and $\bar{i}(j+s) \notin \mathcal{S}'$ for $1 \leq s \leq \ell-1$. However, in order not to count it twice, we don't relate to the segment the contribution from the vertex $\bar{i}(j)$.

We distinguish two cases. If $\ell \geq L(\bar{i}(j+\ell))$ then Lemma 7 implies

$$\left| \prod_{s=j+1}^{j+\ell} \frac{1}{F(\bar{i}(s)) - \lambda} \right| \leq \left(\frac{1}{C_3}\right)^{\ell-1} \left(\frac{2|\lambda|}{\omega}\right)^{-\rho} C_1^\ell. \quad (29)$$

Consider now the case $\ell < L(\bar{i}(j+\ell))$. The possibility $j = 0$ is excluded since this would imply $\ell < |\bar{i}(\ell)_2| \leq \ell$. Thus $\bar{i}(j), \bar{i}(j+\ell) \in \mathcal{S}'$ and necessarily $\bar{i}(j) = \bar{i}(j+\ell)$ as follows from

$$|\bar{i}(j+\ell)_1 - \bar{i}(j)_1| \leq \ell < \text{dist}(\bar{i}(j+\ell)_1, \text{pr}_1(\mathcal{S}) \setminus \{\bar{i}(j+\ell)_1\}).$$

Consequently, ℓ is even. We shall call a segment of this type short loop. To any short loop there exists an opposite short loop $(\bar{\iota}'(j), \bar{\iota}'(j+1), \dots, \bar{\iota}'(j+\ell) = \bar{\iota}'(j))$ defined by $\bar{\iota}'(s) := 2\bar{\iota}(j) - \bar{\iota}(s)$, $j \leq s \leq j+\ell$; hence the base point is the same, $\bar{\iota}'(j) = \bar{\iota}(j)$. Now we are approaching the compensation step. The contribution of two opposite short loops equals

$$\begin{aligned} & \prod_{s=j+1}^{j+\ell} \frac{1}{F(\bar{\iota}(s)) - \lambda} + \prod_{s=j+1}^{j+\ell} \frac{1}{F(\bar{\iota}'(s)) - \lambda} \\ &= \frac{1}{F(\bar{\iota}(j)) - \lambda} \left(\prod_{s=j+1}^{j+\ell-1} \frac{1}{F(\bar{\iota}(s)) - \lambda} - \prod_{s=j+1}^{j+\ell-1} \frac{1}{-F(\bar{\iota}'(s)) + \lambda} \right). \end{aligned} \quad (30)$$

In order to estimate the difference of products on the RHS of (30) one can use the identity

$$u_1 \dots u_N - v_1 \dots v_N = \sum_{s=1}^N u_1 \dots u_{s-1} (u_s - v_s) v_{s+1} \dots v_N \quad (31)$$

and Lemma 9. This way one arrives at

$$|\text{expression(30)}| \leq (\ell-1) \left(\frac{1}{C_3} \right)^{\ell-2} C_2 \left(\frac{2|\lambda|}{\omega} \right)^{-\rho} \leq C_2 C_3^2 \left(\frac{2|\lambda|}{\omega} \right)^{-\rho} \left(\frac{2}{C_3} \right)^{\ell}. \quad (32)$$

In order to treat this type of compensation systematically let us split $\mathcal{P}(M)$ into equivalence classes. Two paths are equivalent if and only if one is obtained from the other by replacing several short loops by their opposites. Thus a path containing s short loops belongs to a class with 2^s elements. One can write schematically

$$\sum_{\text{all paths}} \prod_{\text{all segments}} = \sum_{\text{equivalence classes}} \prod_{\text{pairs of short loops}} \times \prod_{\text{other segments}}$$

For a path $\bar{\iota} \in \mathcal{P}(M)$ denote by $N = N(\bar{\iota})$ the number of vertices belonging to \mathcal{S}' . Obviously, $N(\bar{\iota})$ is constant an every equivalence class. Relying on the estimates (29) and (32) one concludes readily that there exists a constant $\hat{C} > 0$ such that

$$\left| \sum_{\text{equivalence class}} \prod_{j=1}^M \frac{1}{F(\bar{\iota}(j)) - \lambda} \right| \leq \left(\frac{2|\lambda|}{\omega} \right)^{-\rho N} \left(\frac{\hat{C}}{4} \right)^M.$$

Since $b \geq 3$ we have $\bar{u}(1), \bar{u}(2), \bar{u}(3) \notin \mathcal{S}'$ and consequently, as $L(n) \geq 2$ for all $n \in \mathcal{S}'$,

$$2N(\bar{u}) \leq M - 2.$$

To complete the proof it suffices to estimate from above the number of equivalence classes simply by $|\mathcal{P}(M)| \leq 4^M$ (c.f. (28)). ■

With the estimate given in Lemma 10, it is quite straightforward to derive the following existence (but not uniqueness) result.

Lemma 11. *Under the same assumptions as in Lemma 10, the series (27) converges to a solution $g(\beta, \lambda)$ of the equation (6) provided (β, λ) belongs to the domain*

$$\lambda \in \Lambda \cap [-\omega/3, \omega/3], \quad |\beta| \leq (2|\lambda|/\omega)^{\rho/2}/2\hat{C}. \quad (33)$$

For each $\lambda \in \Lambda \cap [-\omega/3, \omega/3]$, $\lambda \neq 0$, the vector-valued function $g(\beta, \lambda)$ is analytic in β on the corresponding neighbourhood of 0 and

$$\|g(\beta, \lambda) + \beta \Gamma_\lambda QVf\| \leq 2\hat{C}^2 \beta^2. \quad (34)$$

Now we can give a precise meaning to the RHS of the fixed-point equation (8). For (β, λ) from the domain (33),

$$G(\beta, \lambda) := \beta \langle QVf, g(\beta, \lambda) \rangle = \sum_{k=1}^{\infty} \beta^{2k} G_{2k}(\lambda),$$

where $G_{2k}(\lambda) := -\langle QVf, (\Gamma_\lambda \hat{V})^{2k-2} \Gamma_\lambda QVf \rangle.$ (35)

In our particular example we have $G_{2k+1}(\lambda) = 0$ for $k \geq 1$ but generally this need not be the case. As a consequence of Lemma 10 we get

$$|G_{2k}(\lambda)| \leq \|V\| \left(\frac{2|\lambda|}{\omega} \right)^\rho \left(\left(\frac{2|\lambda|}{\omega} \right)^{-\rho/2} \hat{C} \right)^{2k-1}. \quad (36)$$

Particularly for our model ($E(1) = 1$),

$$G_2(\lambda) = -\langle QVf, \Gamma_\lambda QVf \rangle = \frac{4(E(1) - \lambda)}{\omega^2 - (E(1) - \lambda)^2},$$

and $G_2(0) \neq 0$.

We shall impose a stricter bound on λ , $|\lambda| \leq \lambda_*$, where $0 < \lambda_* \leq \omega/3$, and we require that λ_* is sufficiently small so that

- $|G_2(\lambda) - G_2(0)| \leq |G_2(0)|/2$,
- $(2\lambda_*/\omega)^{1-\rho} \leq |G_2(0)|/(8\omega\hat{C}^2)$,
- $\lambda_*^{1/2} \leq |G_2(0)|^{3/2}/(16\|V\|\hat{C}^2)$,
- $(2\lambda_*/\omega)^{\rho/2} \leq |G_2(0)|/(2\|V\|\hat{C})$.

Recall that $1/2 < \rho < 1$ (c.f. (16)). Set

$$B(\lambda) := 2(|\lambda|/|G_2(0)|)^{1/2}.$$

The first requirement implies $|G_2(\lambda)| \geq |G_2(0)|/2$ and $\operatorname{sgn} G_2(\lambda) = \operatorname{sgn} G_2(0)$. Owing to the second requirement we have

$$|\lambda| \leq \lambda_* \implies B(\lambda) \leq (2|\lambda|/\omega)^{\rho/2}/2\hat{C}$$

and so $\lambda \in \Lambda \cap [-\lambda_*, \lambda_*]$, $|\beta| \leq B(\lambda)$ determines a subdomain of (33). From the third requirement follows that

$$|\lambda| \leq \lambda_* \implies 2\|V\|\hat{C}^2 B(\lambda)^3 \leq |\lambda|. \quad (37)$$

Finally, a routine calculation based on the definition (35) of G , the estimate (36), and the fourth requirement yields the inequality

$$|\partial_\beta G(\beta, \lambda) - 2\beta G_2(\lambda)| < |\beta| |G_2(0)| \leq 2|\beta| |G_2(\lambda)|, \quad (38)$$

valid for $0 < |\lambda| \leq \lambda_*$, $0 < |\beta| \leq (2|\lambda|/\omega)^{\rho/2}/2\hat{C}$. Consequently,

$$\operatorname{sgn} \partial_\beta G(\beta, \lambda) = \operatorname{sgn} \beta G_2(\lambda) = \operatorname{sgn} \beta G_2(0). \quad (39)$$

Lemma 12. *Under the same assumptions as in Lemma 10, for each $\lambda \in \Lambda \cap [-\lambda_*, \lambda_*]$, $\operatorname{sgn} \lambda = \operatorname{sgn} G_2(0)$, there exist exactly two solutions $\beta_\pm(\lambda)$ to the equation $\lambda = G(\beta, \lambda)$ in the interval $[-B(\lambda), B(\lambda)]$, and there is no solution for $\operatorname{sgn} \lambda = -\operatorname{sgn} G_2(0)$. The two solutions are non-zero, differ in sign, and we choose the convention*

$$-B(\lambda) \leq \beta_-(\lambda) < 0 < \beta_+(\lambda) \leq B(\lambda).$$

Then λ is an eigen-value of the operators $K + \beta_\pm(\lambda) V$.

Remark. Since, in the case of our model, $G(\beta, \lambda)$ is even in β we have consequently $\beta_-(\lambda) = -\beta_+(\lambda)$. But, of course, this is not a general feature.

Proof. Obviously, $G(0, \lambda) = 0$. Let us show that $|G(\pm B(\lambda), \lambda)| \geq |\lambda|$. From (34) we obtain

$$|G(\beta, \lambda) - \beta^2 G_2(\lambda)| = |\beta \langle QVf, g(\beta, \lambda) + \beta \Gamma_\lambda QVf \rangle| \leq 2\|V\| \hat{C}^2 |\beta|^3$$

and, owing to (37),

$$|G(\pm B(\lambda), \lambda) - B(\lambda)^2 G_2(\lambda)| \leq |\lambda|.$$

On the other hand,

$$|B(\lambda)^2 G_2(\lambda)| \geq 4 \frac{|\lambda|}{|G_2(0)|} \cdot \frac{1}{2} |G_2(0)| = 2|\lambda|.$$

This way we have also verified that

$$\operatorname{sgn} G(\pm B(\lambda), \lambda) = \operatorname{sgn} G_2(\lambda) = \operatorname{sgn} G_2(0).$$

Now the existence follows from the fact that the function $G(\beta, \lambda)$ is continuous (even analytic) in β . The uniqueness is a consequence of the monotone behaviour (c.f. (39)). ■

5 Properties of the function $\lambda(\beta)$

Inverting the functions $\beta_+(\lambda)$ and $\beta_-(\lambda)$ we expect to obtain the desired function $\lambda(\beta)$ defined respectively on sets I_+ and I_- , with $I_\pm \subset \mathbb{R}_\pm$, and we set naturally $\lambda(0) = 0$. Thus the total domain for $\lambda(\beta)$ is $I = I_- \cup \{0\} \cup I_+$. $\lambda(\beta)$ is positive (negative), except of $\lambda(0) = 0$, if $G_2(0)$ is positive (negative). The existence of the inverted function follows from the monotone behaviour of the original functions $\beta_\pm(\lambda)$.

We shall need

Lemma 13. *The function $G(\beta, \lambda)$ defined in (35) fulfills the equality*

$$G(\beta, \lambda_2) - G(\beta, \lambda_1) = -(\lambda_2 - \lambda_1) \langle g(\beta, \lambda_2), g(\beta, \lambda_1) \rangle$$

for all

$$\lambda_1, \lambda_2 \in \Lambda \cap [-\omega/3, \omega/3], \quad |\beta| \leq (2 \min\{|\lambda_1|, |\lambda_2|\}/\omega)^{\rho/2} / 2\hat{C}. \quad (40)$$

Proof. Note that $\Gamma_{\lambda_2} - \Gamma_{\lambda_1} = (\lambda_2 - \lambda_1) \Gamma_{\lambda_2} \Gamma_{\lambda_1}$ on $\mathcal{D}(\Gamma_{\lambda_1}) \cap \mathcal{D}(\Gamma_{\lambda_2})$ and consequently, using (31),

$$\begin{aligned} & \langle QVf, (\Gamma_{\lambda_2} \hat{V})^k \Gamma_{\lambda_2} QVf - (\Gamma_{\lambda_1} \hat{V})^k \Gamma_{\lambda_1} QVf \rangle \\ &= (\lambda_2 - \lambda_1) \sum_{j=0}^k \langle (\Gamma_{\lambda_2} \hat{V})^j \Gamma_{\lambda_2} QVf, (\Gamma_{\lambda_1} \hat{V})^{k-j} \Gamma_{\lambda_1} QVf \rangle. \end{aligned}$$

Now the identity can be verified easily with the aid of (27). ■

From (34) one deduces that $\langle g(\beta, \lambda_2), g(\beta, \lambda_1) \rangle > 0$ whenever $|\lambda_1|, |\lambda_2|$ are sufficiently small and $|\beta|$ obeys (40). Thus we find that $G(\beta, \lambda)$ is strictly decreasing in λ for every β fixed. The same is true for the function $\Phi(\beta, \lambda) := G(\beta, \lambda) - \lambda$.

This is an elementary exercise to verify that the functions $\beta_{\pm}(\lambda)$ are strictly monotone provided one uses the equality $\Phi(\beta_{\pm}(\lambda), \lambda) = 0$ and the fact that $\Phi(\beta, \lambda)$ is monotone in β (c.f. (39)) and strictly monotone in λ . We can formulate our conclusion as follows.

Lemma 14. *There exists a bound λ_{**} , $0 < \lambda_{**} \leq \lambda_*$, and a function $\lambda(\beta)$ defined on $I \subset \mathbb{R}$ such that $0 \in I$ and $\lambda(0) = 0$, $\beta_{\pm}(\lambda(\beta)) = \beta$ for $\forall \beta \in I \cap \mathbb{R}_{\pm}$, and the range of both $\lambda(\beta)|_{I \cap \mathbb{R}_+}$ and $\lambda(\beta)|_{I \cap \mathbb{R}_-}$ equals either $\Lambda \cap [0, \lambda_{**}]$ or $\Lambda \cap [-\lambda_{**}, 0]$ depending on whether $G_2(0)$ is positive or negative. For $\forall \beta \in I$, $\lambda(\beta)$ is an eigen-value of the operator $K + \beta V$.*

This seems to be a typical feature for the perturbation theory of dense point spectra that one has to abandon some values of the coupling constant β and to determine the perturbed eigen-value as a function $\lambda(\beta)$ defined on a domain I possessing "holes". To treat functions of this type one can refer to the celebrated Whitney Extension Theorem (see Stein 1970). In fact, its proof in the one-dimensional case is rather elementary. We shall need the following very particular version.

Lemma 15. *Let χ be a real function defined on a closed subset $Y \subset \mathbb{R}$, χ is monotone, and suppose that there exist two constants $0 < A \leq B$ such that*

$$A|y_1 - y_2| \leq |\chi(y_1) - \chi(y_2)| \leq B|y_1 - y_2| \quad \text{for all } y_1, y_2 \in Y.$$

Then there exists an extension $\tilde{\chi}$ defined on \mathbb{R} , $\tilde{\chi}|_Y = \chi$, and $\tilde{\chi}$ is again monotone and obeys the same inequalities but this time on the whole line \mathbb{R} ,

$$A|y_1 - y_2| \leq |\tilde{\chi}(y_1) - \tilde{\chi}(y_2)| \leq B|y_1 - y_2| \quad \text{for all } y_1, y_2 \in \mathbb{R}.$$

Proof. The complement of Y is an open subset of \mathbb{R} and hence at most countable disjoint union of open intervals. One defines the function $\tilde{\chi}$ linearly on these intervals requiring it to be continuous. Provided the interval in question is half-infinite then $\tilde{\chi}$ is defined again linearly with the slope lying between A and B . The inequalities for $\tilde{\chi}$ defined this way are easy to verify; for the left one we need that χ is monotone. ■

We wish to show that 0 is a point of density for the set I . We already know that this is true for the set Λ (Lemma 6). The intermediate step is given by

Lemma 16. *Assume that a real function $\varphi(x)$, defined on a set $X \subset [0, +\infty[$, is strictly increasing, $\varphi(0) = 0$ ($\Rightarrow 0 \in X$), and the set $Y = \varphi(X)$ is closed. Moreover, suppose that there exist two constants $0 < A \leq B$ such that*

$$A|x_1^2 - x_2^2| \leq |\varphi(x_1) - \varphi(x_2)| \leq B|x_1^2 - x_2^2| \quad \text{for all } x_1, x_2 \in X. \quad (41)$$

Then it holds

$$\lim_{\eta \downarrow 0} |Y \cap [0, \eta]|/\eta = 1 \implies \lim_{\delta \downarrow 0} |X \cap [0, \delta]|/\delta = 1. \quad (42)$$

Proof. Apply Lemma 14 to the function $\chi(y) = (\varphi^{-1}(y))^2$ (the corresponding constants are $0 < 1/B \leq 1/A$). The extension $\tilde{\chi}$ is again strictly increasing, $\tilde{\chi}(y) > 0$ for $y > 0$, and $\tilde{\chi}(\mathbb{R}_+) = \mathbb{R}_+$. Define $\tilde{\varphi}$ on \mathbb{R}_+ by $\tilde{\varphi}(x) = y$ iff $x^2 = \tilde{\chi}(y)$, i.e., $\tilde{\varphi}$ is the inverse of $(\tilde{\chi}|_{\mathbb{R}_+})^{1/2}$. Clearly, the function $\tilde{\varphi}$ is an extension of φ , $\tilde{\varphi}|_X = \varphi$, it is again strictly increasing, and the inequalities (41) hold for $\tilde{\varphi}$ on the whole positive half-line. Consequently, $\tilde{\varphi}$ is absolutely continuous on every bounded interval, $\tilde{\varphi}'$ exists almost everywhere, and it holds

$$\tilde{\varphi}(x) \leq Bx^2 \text{ and } 2Ax \leq \tilde{\varphi}'(x) \quad \text{for (almost) all } x \geq 0.$$

Denote by X^c and Y^c the complements of X and Y in $[0, +\infty[$, respectively. The implication (42) is equivalent to

$$\lim_{\eta \downarrow 0} |Y^c \cap [0, \eta]|/\eta = 0 \implies \lim_{\delta \downarrow 0} |X^c \cap [0, \delta]|/\delta = 0. \quad (43)$$

Choose p , $1 < p < 2$, and let q be the adjoint exponent, $p^{-1} + q^{-1} = 1$. We shall verify the inequality

$$\delta^{-1} |X^c \cap [0, \delta]| \leq \frac{B}{2A} \left(1 - \frac{p}{2}\right)^{-1/p} \left(\tilde{\varphi}(\delta)^{-1} |Y^c \cap [0, \tilde{\varphi}(\delta)]|\right)^{1/q}. \quad (44)$$

It is clear that (43) is a consequence of (44). We have

$$|X^c \cap [0, \delta]| = \int_{Y^c \cap [0, \tilde{\varphi}(\delta)]} \frac{dy}{\tilde{\varphi}'(\tilde{\varphi}^{-1}(y))} \leq \frac{\sqrt{B}}{2A} \int_{Y^c \cap [0, \tilde{\varphi}(\delta)]} y^{-1/2} dy$$

since $\tilde{\varphi}'(\tilde{\varphi}^{-1}(y)) \geq 2A \tilde{\varphi}^{-1}(y) \geq 2A(y/B)^{1/2}$. Hölder Inequality then gives

$$\int_{Y^c \cap [0, \tilde{\varphi}(\delta)]} y^{-1/2} dy \leq \left(\int_0^{\tilde{\varphi}(\delta)} y^{-p/2} dy \right)^{1/p} \left(\int_{Y^c \cap [0, \tilde{\varphi}(\delta)]} dy \right)^{1/q}$$

and (44) follows immediately. ■

Observe that the property (2) given in Proposition 1 is equivalent to

$$\lim_{\delta \downarrow 0} |I \cap [0, \delta]|/\delta = 1 \quad \text{and} \quad \lim_{\delta \downarrow 0} |I \cap [-\delta, 0]|/\delta = 1.$$

Thus we can treat the right and the left neighbourhood of 0 separately. We can now apply Lemma 16 to the function $\lambda(\beta)$ instead of $\varphi(x)$ and to the sets $I_+ \cup \{0\}$ and $I_- \cup \{0\}$ instead of X . Observe from the definition (14) that Λ is closed. Let us show that the condition (41) is fulfilled as well. Assume that $\beta_1, \beta_2 \in I$, $|\beta_1| < |\beta_2|$. Then $(\beta_1, \lambda(\beta_1))$, $(\beta_2, \lambda(\beta_2))$ and $(\beta_1, \lambda(\beta_2))$ belong to the domain of G . Write

$$\lambda(\beta_1) - \lambda(\beta_2) = G(\beta_1, \lambda(\beta_1)) - G(\beta_1, \lambda(\beta_2)) + G(\beta_1, \lambda(\beta_2)) - G(\beta_2, \lambda(\beta_2))$$

and use Lemma 13 to get

$$\lambda(\beta_1) - \lambda(\beta_2) = (G(\beta_1, \lambda(\beta_2)) - G(\beta_2, \lambda(\beta_2))) / (1 + \langle g(\beta_1, \lambda(\beta_1)), g(\beta_1, \lambda(\beta_2)) \rangle).$$

Deduce from (34) that

$$0 < \langle g(\beta_1, \lambda(\beta_1)), g(\beta_1, \lambda(\beta_2)) \rangle = O(|\beta_2|^2), \text{ as } |\beta_1| \leq |\beta_2| \rightarrow 0,$$

and note that (38) can be rewritten as

$$|\partial_{\beta^2} G(\beta, \lambda) - G_2(\lambda)| \leq |G_2(0)|/2.$$

One readily concludes that there exist constants $0 < A \leq B$ and a bound $\beta_\star > 0$ such that

$$A|\beta_1^2 - \beta_2^2| \leq |\lambda(\beta_1) - \lambda(\beta_2)| \leq B|\beta_1^2 - \beta_2^2| \quad \text{for all } \beta_1, \beta_2 \in I \cap [-\beta_\star, \beta_\star].$$

Lemma 17. *0 is a point of density for the set I .*

Now we can approach the problem of the asymptotic series. Consider first the following situation. Let $\{H_k\}_{k=0}^\infty$ be a sequence of complex meromorphic functions such that $H'_0(0) \neq 0$ and 0 is a regular point for all of them. Then

$$\Phi(x, y) := \sum_{k=0}^{\infty} x^k H_k(y) \in \mathbb{C}[[x, y]]$$

is well defined as a formal power series in x and y . Denote by $\varphi^f(x) \in \mathbb{C}[[x]]$ the solution to the problem

$$\varphi^f(0) = 0 \quad \text{and} \quad \Phi(x, \varphi^f(x)) = 0,$$

which exists and is unique in the class of formal power series. Set

$$\mathcal{R}_\Phi := \mathbb{C} \setminus \bigcup_{k=0}^{\infty} \{\text{the poles of the function } H_k\}$$

and let $R(y)$ be the radius of convergence of the series $\Phi(x, y)$ in the variable x , with $y \in \mathcal{R}_\Phi$ being fixed.

Lemma 18. *Let φ be a complex function defined on $X \subset \mathbb{C}$ and assume that:*

- (1) $0 \in X$ is an accumulation point of X ,
- (2) $\forall x \in X, |x| < R(\varphi(x))$ (and so the value $\Phi(x, \varphi(x))$ is well defined),
- (3) φ solves the problem

$$\varphi(0) = 0 \quad \text{and} \quad \Phi(x, \varphi(x)) = 0 \quad \text{for } \forall x \in X,$$

- (4) there exists $\mu > 0$ such that

$$\begin{aligned} \Phi_N(x, \varphi(x)) &= O(|x|^{\mu(N+1)}) \quad \text{for } \forall N \in \mathbb{Z}_+, \text{ where} \\ \Phi_N(x, y) &:= \sum_{k=0}^N x^k H_k(y). \end{aligned}$$

Then $\varphi^f(x)$ is an asymptotic series for $\varphi(x)$.

Proof. Denote by φ_M^f the truncation of φ^f (thus φ_M^f is a polynomial of degree at most M and $\varphi^f(x) - \varphi_M^f(x) \in x^{M+1} \mathbb{C}[[x]]$). We have to show that

$$\varphi(x) - \varphi_M^f(x) = O(|x|^{M+1}), \quad \forall M \in \mathbb{Z}_+.$$

Denote by $\varphi^{(N)}(x)$ the unique solution to the problem

$$\varphi^{(N)}(0) = 0 \quad \text{and} \quad \Phi_N(x, \varphi^{(N)}(x)) = 0,$$

in the class of germs of holomorphic functions at $x = 0$. Clearly,

$$\varphi_M^f(x) = \varphi_M^{(N)}(x) \quad \text{if } N \geq M.$$

Note that the requirement (4), with $N = 0$, means that $H_0(\varphi(x)) = O(|x|^\mu)$. Since $H'_0(0) \neq 0$ we find that $\lim_{x \rightarrow 0} \varphi(x) = 0$. Obviously, it also holds that $\lim_{x \rightarrow 0} \varphi^{(N)}(x) = 0$. Observe that $\partial_y \Phi_N(0, 0) = H'_0(0) \neq 0$. Consequently, for any $n \in \mathbb{Z}_+$, there exist positive constants c_N, δ_N such that

$$|\Phi_N(x, \varphi(x)) - \Phi_N(x, \varphi^{(N)}(x))| \geq c_N |\varphi(x) - \varphi^{(N)}(x)| \quad \text{for } \forall x \in X, |x| \leq \delta_N.$$

Fix $M \in \mathbb{Z}_+$ and choose $N \in \mathbb{Z}_+$ such that $N \geq M$ and $\mu(N+1) \geq M+1$. Write

$$\varphi(x) - \varphi_M^f(x) = \varphi(x) - \varphi^{(N)}(x) + \varphi^{(N)}(x) - \varphi_M^{(N)}(x) = \varphi(x) - \varphi^{(N)}(x) + O(|x|^{M+1}).$$

On the other hand,

$$c_N |\varphi(x) - \varphi^{(N)}(x)| \leq |\Phi_N(x, \varphi(x)) - \Phi_N(x, \varphi^{(N)}(x))| = |\Phi_N(x, \varphi(x))| = O(|x|^{\mu(N+1)}).$$

We conclude that $\varphi(x) - \varphi_M^f(x) = O(|x|^{M+1})$, as required. ■

Lemma 17 is directly applicable to the function $\Phi(\beta, \lambda) := G(\beta, \lambda) - \lambda$ and to our solution $\lambda(\beta)$.

Lemma 19. *The formal power series $\sum_{M=0}^{\infty} \xi_M \beta^M$, with ξ_M given in (10) and (13), is an asymptotic series for the function $\lambda(\beta)$ defined on I .*

Let us summarize that Lemma 14, Lemma 17 and Lemma 19 verify jointly the existence and the properties of the function $\lambda(\beta)$ and thus the proof of Proposition 1 has been completed.

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References

- [1] Bellissard J 1985 *Trends and Developments in the Eighties* Albeverio and Blanchard Eds. (Singapore: Word Scientific) pp. 1-106
- [2] Bellissard J and Vittot M 1990 *Ann. Inst. Henri Poincaré* **52** 175-235
- [3] Bleher P M, Jauslin H R, Lebowitz J L 1992 dependent *J. Stat. Phys* **68** 271
- [4] Combescure M 1987 *Ann. Inst. Henri Poincaré* **47** 62-82
- [5] Duclos P and Šťovíček P 1996 *Commun. Math. Phys.* **177** 327-247
- [6] Eliasson L H 1988 *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **4** 15, no. 1, 115-147
- [7] Howland J S 1979 *Indiana J. Math.* **28** 471-494
- [8] Joye A *J. Stat. Phys.* **75** 929-952
- [9] Kato T 1966 *Perturbation theory of linear operators* (New York: Springer-Verlag)
- [10] Nenciu G 1993 *Ann. Inst. Henri Poincaré* **59** 91-97
- [11] Rellich F 1937 *Math. Ann.* **113** 600-619
- [12] Simon B 1993 *Spectral analysis of rank one perturbations and applications* (lecture given at the Vancouver Summer School in Mathematical Physics, August 10-14)
- [13] Stein E M 1970 *Singular Integrals and Differentiability Properties of Functions* (New Jersey: Princeton University Press)
- [14] Yajima K 1977 *J. Math. Soc. Japan* **29** 729-743